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THE CIRCULAR NUMBERS FOR A PLANE CURVE.

BY HORACE T. BURGESS.*

Introduction.—The well-known numbers of Plücker for a plane curve are arithmetical invariants of the curve under the transformations of the projective group in the plane. The object of this paper is to derive a set of numbers for a plane curve such that each is an arithmetical invariant of the curve under the transformations † of the circular group in the plane; to establish the equations connecting these numbers; and to illustrate by some simple examples.

1. **Derivation.**—Let γ be a plane curve; project γ stereographically upon a sphere Σ and denote the projection by Γ . Any projective transformation of space leaving Σ invariant is equivalent to a transformation of the circular group in the plane of γ .‡ Any numbers which are arithmetical invariants of Γ under the projective group G_6 , which leaves Σ invariant, must be invariants of γ under the circular G_6 in the plane of γ . We shall choose as our invariants of Γ under the projective G_6 Cayley's numbers§ $m, r, n, \alpha, \beta, x, y, g, h$, together with the five numbers d, i, j, p, f , defined as follows: We shall take d to be the number of actual double points upon Γ ; if we refer to the two systems of generators upon Σ as the i -system and the j -system, then i and j are the number of points of Γ in common with each generator of the i -system and j -system respectively; and p and f , we define by the invariant equations $2p = i(i - 1) + j(j - 1)$ and $f = (r - 2i)(r - 2j)$. We have also $m = i + j$.||

Let O be the center of projection on Σ . If γ meets the line at infinity, IJ , only in the circular points I and J , Γ does not pass through O . Excluding the circular points I and J : If γ intersects IJ in t distinct points, Γ has t distinct branches through O ; if γ touches IJ , Γ has a cusp at O ; if γ has contact of higher order with IJ , Γ has a singularity of higher order at O . A singularity arising at O on Γ in this manner is equivalent to a certain number of nodes and cusps,** and these in general may be included

* Presented to the American Mathematical Society (Minneapolis, 1910).

† Newson, Bulletin of the American Mathematical Society, Dec., 1897.

‡ Klein, Einleitung in die höhere Geometrie, I, pp. 378–380.

§ Salmon, Solid Geometry, 4th ed., pp. 291–5.

|| Clebsch-Lindemann, Vorlesungen über Geometrie, Band II, p. 418.

** Charlotte Scott, American Journal of Mathematics, Vol. 14, pp. 301–325. The singularity arising on Γ at O due to a given order of contact of γ with IJ is of the same type as that arising in quadric inversion in the plane with OIJ as the fundamental triangle.

in the numbers d and β for Γ such that Cayley's equations* are satisfied. Those plane curves, which on projection give rise to singularities at O so complicated that Cayley's equations are not satisfied in the above manner, must be excluded from our consideration. Since Γ lies entirely upon Σ , each one of Cayley's numbers for Γ has a particular significance. This is so obvious geometrically that we shall state it briefly in the following parallel columns which are self-explanatory:

Cayley's Numbers for Γ .

- m : The degree of Γ is the number of points of Γ in an arbitrary plane E .
- r : The rank of Γ is the number of tangents of Γ cutting an arbitrary line L .
- n : The class of Γ is the number of osculating planes of Γ through an arbitrary point P .
- α : The number of stationary planes of Γ .
- β : The number of cusps of Γ .
- d : The number of nodes of Γ .
- x : The number of points on two tangents of Γ which lie in an arbitrary plane E .
- y : The number of planes through two tangents of Γ which also pass through an arbitrary point P .
- g : The number of lines in two osculating planes of Γ which lie in an arbitrary plane E .

Cayley's Numbers for Γ upon Σ .

- The number of intersections of Γ with the circle cut out of Σ by E .
- The number of circles, cut out of Σ by the pencil of planes through L , which touch Γ .
- The number of osculating circles of Γ cut out of Σ by the bundle of planes through P .
- When P is upon Σ , the circles all pass through P .
- The number of stationary circles of Γ .
- The number of cusps of Γ .
- The number of nodes of Γ .
- The number of doubly orthogonal circles of Γ cut out of Σ by the bundle of planes through the pole of E with respect to Σ .
- When E is tangent to Σ at the point P , the circles all pass through P .
- The number of bitangent circles of Γ cut out of Σ by the bundle of planes through P .
- When P is upon Σ , the circles all pass through P .
- The number of pairs of osculating circles of Γ which intersect on the circle cut out of Σ by E .

* Cf. Salmon, loc. cit., p. 295.

- h : The number of lines through two points of Γ and an arbitrary point P . This includes both the actual and apparent double points of Γ .
- The number of pairs of tangent circles of Γ cut out of Σ by the planes through P such that each circle passes through the point of contact of the other.

p : It will be shown later that p is the number of apparent double points of Γ upon Σ .

f : If r is the rank of Γ , then any line L , which intersects Γ in κ distinct points, is met by $r - 2\kappa$ tangents of Γ excluding the κ tangents at the κ points common to Γ and L . Further if L is a generator of Σ belonging to the i -system, it meets Γ in i points, and L is met by $r - 2i$ tangents of Γ which are generators of the j -system; for all of the tangents of Γ which meet a generator of Σ are generators of Σ . Hence any generator of the i -system is met by $r - 2i$ generators of the j -system which are tangent to Γ ; and any generator of the j -system is met by $r - 2j$ generators of the i -system which are tangent to Γ . These $r - 2i$ generators of the j -system and $r - 2j$ generators of the i -system meet in f points.

To get the geometrical significance of these invariants of γ , we project Γ back stereographically into γ together with the configurations we have just stated. We shall use the corresponding capital letters to denote these projected configurations in the plane. Singularities on Γ which are not at the center of projection O project into the same singularities on γ . If Γ has a singularity at O , on projection it breaks up into intersections and contacts of various orders with the line IJ ; and it follows that B and D corresponding respectively to β and d are not the number of nodes and cusps on γ . B we shall call the *cuspidal equivalence* of γ ; and in like manner, excluding the nodes at I and J , D we shall call the *nodal equivalence* of γ . The generators of Σ project into the minimum lines of the plan; and if the i -system projects into the I -pencil, the j -system projects into the J -pencil. The projecting lines OI and OJ are generators of Σ ; OI belongs to the j -system and OJ belongs to the i -system; the f points, in which the $r - 2i$ generators tangent to Γ and meeting OJ , and the $r - 2j$ generators tangent to Γ and meeting OI intersect, project into the foci of γ .

The significance of the other projected configurations seems to be so obvious that we state without further consideration the following:

THEOREM I. *The plane curve γ has the following invariant numbers and geometrical configurations under the transformations of the circular group G_6 :*

M : The circular degree of γ is the number of points of γ in common with an arbitrary circle excluding the circular points I and J .

R : The circular class of γ is the number of circles of an arbitrary pencil touching γ .

N : The circular inflexion of γ is the number of osculating circles of γ through an arbitrary point.

A : The number of stationary circles of γ .

B : The cuspidal equivalence of γ .

D : The nodal equivalence of γ excluding the nodes at I and J .

X : The number of doubly orthogonal circles of γ through an arbitrary point.

Y : The number of bitangent circles of γ through an arbitrary point.

G : The number of pairs of osculating circles of γ intersecting upon an arbitrary circle.

H : The number of pairs of tangent circles of γ cutting an arbitrary circle orthogonally while each circle of a pair passes through the point of contact of the other.

I : The number of points of γ in common with a minimum line through I , excluding I .

J : The number of points of γ in common with a minimum line through J , excluding J .

F : The number of foci of γ .

2. **Circular Curves.**—Those plane curves which meet the line IJ only in the circular points, we call *circular curves*. If γ is a *circular curve*, Γ does not pass through O and the cone which projects Γ into γ shows $M = \mu$, $R = \nu$, $N = i$, $B = \kappa$, $Y = \tau$, $H = \delta$, where μ , ν , etc., are Plücker's numbers for γ .* Further, in our definition of x , if we choose the arbitrary plane E tangent to Σ at O , all the doubly orthogonal circles pass through O and each projects into a binormal of γ . Likewise in our definition of g , if we take the arbitrary plane E through O , we have, on projecting, the number of pairs of osculating circles of γ having a common radical axis. P is the number of nodes at I and J ; for γ has J branches through I and I branches through J .

These considerations enable us to state the following theorem for *circular curves* in the plane:

THEOREM II. *If γ is a circular curve, it has the following geometrical properties:*

$M = \mu$, the Plücker degree.

$R = \nu$, the Plücker class.

$N = i$, the number of inflexions.

A = the number of stationary circles.

$B = \kappa$, the number of cusps.

D = the number of nodes, excluding those at I and J .

X = the number of binormals.

$Y = \tau$, the number of bitangents.

* Cf. Salmon, loc. cit., p. 295.

G = the number of pairs of osculating circles having a common radical axis.

$H = \delta$, the number of nodes.

I = the number of branches through J .

J = the number of branches through I .

P = the number of nodes at I and J .

F = the number of foci.

It is obvious that any plane curve may be transformed into a *circular* curve by a transformation of the circular group G_6 ; for we have only to transform Γ by a rotation of Σ such that the transformed curve on Σ does not pass through O and the equivalent transformation of the circular group transforms γ into a *circular* curve. We have then from Theorem II the particular significance of these numbers for the *circular* curve into which any *noncircular* curve may be transformed by a transformation of the circular group G_6 .

3. **Equations.**—The circular numbers satisfy the following equations:

$$(1) \quad M = 3N(N - 2) - 6G - 8A,$$

$$(2) \quad M = R(R - 1) - 2Y - 3N,$$

$$(3) \quad R = N(N - 1) - 2G - 3A,$$

$$(4) \quad R = M(M - 1) - 2H - 3B,$$

$$(5) \quad N = R(R - 1) - 2X - 3M,$$

$$(6) \quad N = 3M(M - 2) - 6H - 8B,$$

$$(7) \quad A = 3R(R - 2) - 6X - 8M,$$

$$(8) \quad B = 3R(R - 2) - 6Y - 8N,$$

$$(9) \quad P = \frac{1}{2}[I(I - 1) + J(J - 1)],$$

$$(10) \quad F = (R - 2I)(R - 2J),$$

$$(11) \quad M = I + J,$$

$$(12) \quad H = D + P.$$

The first eight of these equations are taken directly from those of Cayley for Γ ; hence, when any three of the circular numbers $M, R, N, A, B, X, Y, G, H$ are known, the remaining six may be found. The last equation follows directly from Theorem II, and shows p to be the number of apparent double points of Γ on Σ . The other three equations follow from those already stated for space.

4. **Equivalence.** DEFINITION: Two plane curves are equivalent with

respect to the circular group provided they are transforms of one another by transformations of this group.

THEOREM III. *A necessary condition for the equivalence of two plane curves with respect to the circular group is that they have the same circular numbers.*

The truth of this theorem follows directly from the above definition and the invariance of the circular numbers. That this theorem is not sufficient will be shown by one of the following examples.

5. **Examples.**—In general, to find the circular numbers for a plane curve γ , we find B, D, I , and J from which we calculate the others by means of our equations. In this manner we find the characteristics of the familiar curves as given in the following table:

Curve.....	M	R	N	A	B	D	X	Y	G	H	I	J	P	F
Parabola.....	4	5	4	1	1	0	2	2	2	2	2	2	2	1
Ellipse *.....	4	6	6	4	0	1	6	4	6	3	2	2	2	4
Witch.....	6	10	12	12	0	4	30	24	43	10	3	3	6	16
Cassini's oval.....	4	8	12	16	0	0	16	8	38	2	2	2	2	16
$y^2 = ax^3$	6	7	6	3	3	1	9	9	7	7	3	3	6	1

The parabola, cissoid, and cardioid are equivalent† and hence have the same circular numbers by Theorem III. The ellipse, hyperbola, lemniscate, limaçon, and strophoid are found, in the above-mentioned manner, to possess the same circular numbers; likewise the witch, folium, and three-leaved rose are found to have the same circular numbers. The ellipse and hyperbola are not equivalent for there exists no transformation of our G_6 which will transform the one into the other; this example shows that Theorem III is not sufficient. The ellipse and the limaçon with the conjugate point are equivalent;‡ the hyperbola and the limaçon with the node are equivalent.§ The equilateral hyperbola, lemniscate, and strophoid are equivalent.¶

The curve $y = x^3$ furnishes the simplest example of a case where the circular numbers do not satisfy our equations. This curve has a cusp on the line at infinity.

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* Cf. Klein, *Math. Annalen*, Vol. 10, p. 208.

† Cf. Smith and Gale, *Analytic Geometry*, pp. 299, 302.

‡ Cf. Smith and Gale, *loc. cit.*, p. 302.

§ Cf. Smith and Gale, *loc. cit.*, pp. 300, 301.